HANDOUT 3:
HOW SET THEORY CAPTURES IMPORTANT PROPERTIES AND RELATIONS

The starting point for our semantic theory was the assumption that sentence denotations can be identified with sets of possible situations. This move gave us a first handle on a simple case of compositionality. By identifying the denotation of and with set intersection, and the denotation of or with set union, we were able to compute the meanings of complex sentences built with the help of **and** and **or**. We might also note that when we negate a proposition, we map it into its complement. I didn't write down a denotation for not, because in English, you don't negate a sentence by simply putting not in front of it. Negation, then, requires a little more thought, as far as the relation between syntactic structures and semantic interpretation is concerned. If we take sentence denotations to be sets of possible situations, we can easily define the most important semantic properties and relations

Consistency and Logical Consequence

Among the most important semantic notions are the notions 'consistency' and 'logical consequence'. Judgements about consistency and inconsistency are ever present in our lives. An inconsistent testimony disqualifies a witness. Inconsistency is the worst verdict for any theory. The notion of 'logical consequence' (= 'logical implication') is equally important. When you make a claim, you are committed to what it logically implies. If sentence denotations are sets of possible situations, it's easy and straightforward to define the notions 'consistency' and 'logical consequence'.

Consistency (informal)

A set of sentences is consistent, if and only if the members of the set could all be true together in some possible situation.

Logical consequence (informal)

A set of sentences $A = \{ \alpha_1 \ldots\alpha_n\}$ logically implies a sentence β if and only if there is no possible situation in which all members of A are true, but β is false.

Class exercise 1

Which of the following sets of sentences are consistent? Surprisingly, we are all able to work out the answers to this question, given a little time. This is a major achievement if you think about it.

Example 1 (W. Hodges)
During the last five years I have been involved in three major accidents and several minor ones, while driving my car. After two of the major accidents, courts held me responsible. But basically I'm a thoroughly safe driver; I've simply had a run of bad luck.

Example 2 (W. Hodges)
The surface of the earth is flat (apart from mountains, oceans and other relatively small bumps and dips). When people think they have sailed round the earth, all they have done is to set out from one place and finish up in another place exactly like the one they started from, but several thousand miles away.

Example 3 (W. Hodges)
I have invented an amazing new sedative which makes people faster and more excited.

Example 4 (W. Hodges)
I've never drawn anything in my life. But if I sat down to it now, it would take me two minutes to produce a drawing worth as much as anything Picasso has produced.

Example 5 (W. Hodges)
There is no housing shortage in Lincoln today - just a rumor that is put about by people who have nowhere to live.

Class exercise 2

Determine whether the set of premises logically implies the conclusion in the following examples.

Example 1 (W. Hodges)

Premises
(1) My car doesn't start.
(2) When Jones's car didn't start, the trouble was a wet distributor.
(3) Cars like mine have this as a common problem.
(4) It is very damp today.

Conclusion
(5) The fault in the car is a wet distributor.

Example 3 (W. Hodges)

Premises
(1) If Jones died of natural causes and he had been to the doctor recently, then that doctor was negligent.
(2) Jones had been to that doctor.
(3) That doctor was found guilty of negligence.

Conclusion
(4) Jones died of natural causes.

Example 4 (W. Hodges)

Premises
(1) The Earth is either round or not round.
(2) If the Earth is round, then many things would just fall off it.
(3) Things do not just fall off the Earth.

Conclusion
(4) The Earth is not round.

Class Exercise 3

Find a more formal version of consistency

Consider a set of sentences $\{ \alpha_1 \ldots . \alpha_n \}$. State a set-theoretic condition that $\{ [[\alpha_1]] \ldots .[[\alpha_n]] \}$ has to satisfy for $\{ \alpha_1 \ldots . \alpha_n \}$ to be consistent.

Hints:

Remember that for any sentence α, $[[\alpha]]$ is the set of possible situations in which α is true.

It might also help to remember our informal version of consistency. We said that a set of sentences is consistent, if and only if the members of the set could all be true together in some possible situation.

Can we get the set of all possible situation s in which the sentences α_1 through α_n are all true?

What do we know about that set?

Find a more formal definition of logical consequence

State in set-theoretic terms how $\{ [[\alpha_1]] \ldots .[[\alpha_n]] \}$ and $[[\beta]]$ have to be related for the set of sentences $\{ \alpha_1 \ldots . \alpha_n \}$ to logically imply the sentence β.

Hints:

Again, recall our informal version. A set of sentences $A = \{ \alpha_1 \ldots . \alpha_n \}$ logically implies a sentence β if and only if there is no possible situation in which all members of A are true, but β is false. This is the same as saying that all of the situations in which α_1 through α_n are true are situations in which β is true.

Can you collect all of the situations in which α_1 through α_n are is true into a set?

Can you collect all of the situations in which β is true into a set?

How are these two sets related?

Formal definitions to ~~remember~~ <u>understand</u>

A sentence α is a <u>contradiction</u> iff $[[\alpha]] = \emptyset$.

A sentence α is a <u>tautology</u> iff $[[\alpha]] = S$.

A sentence α describes a <u>logical possibility</u> iff $[[\alpha]] \neq \emptyset$.

A sentence α <u>logically implies</u> a sentence β iff $[[\alpha]] \subseteq [[\beta]]$. (Instead of saying "$\alpha$ logically implies β", you can also say "β follows from α")

Note: assume $n \geq 2$ in the following two definition.

A set of sentences $\{ \alpha_1 \ldots\ldots \alpha_n \}$[i] is <u>consistent</u> iff $[[\alpha_1]] \cap [[\alpha_2]] \ldots\ldots \cap [[\alpha_n]] \neq \emptyset$.

A set of sentences $\{ \alpha \ldots\ldots \alpha_n \}$ <u>logically implies</u> a sentence β iff $[[\alpha\ ]] \cap [[\alpha\ ]] \ldots\ldots \cap [[\alpha\ ]] \subseteq [[\beta]]$. Here, too, you can say "β <u>follows from</u> $\{ \alpha \ldots\ldots \alpha\ \}$" instead of "$\{ \alpha \ldots\ldots \alpha \quad \}$ <u>logically implies</u> β ".

Reflections

The semantic properties and relations I have just mentioned were defined as properties of sentences and relations between sentences. However, all of those definitions can be easily reformulated so as to give us properties of propositions and relations between propositions. Here are some statements about propositions that you should understand without difficulties. Talk to me if there is anything that is still unclear to you. Suppose p and q are propositions (that is, sets of possible situations), the proposition p logically implies q iff p is a subset of q. If p logically implies q, but not the other way round, then p is a <u>proper</u> subset of q. If p is a proper subset of q, then p is <u>more informative</u> or <u>stronger</u> than q; In this case, if I assert p, I'm claiming that the actual situation is a member of a smaller set of possible situation than if I were to assert q. The set of all possible situations S is the <u>least</u> <u>informative</u>, or weakest proposition. Among all the possible situations in S, some and only some are <u>actual</u>. Those are the situations that are part of the actual world. When I say that I have a cat, for example, my assertion is <u>true</u> just in case the proposition I expressed is <u>true</u> in some actual situation. A proposition p is <u>true</u>, then, iff there is an actual situation that is a member of p.

Practice exercises

To test your understanding of the material in this handout, work through the following exercises.

a. Look for examples of English sentences that are tautologies or contradictions.

b. Find a set A of three English sentences that has the following properties:
A is not consistent, but each proper subset of A is consistent.

c. Show that the arbitrarily picked sentence "The moon is orange" logically implies any tautology, and is logically implied by any contradiction.

d. Find a set A of three sentences that has the following properties: A logically implies "The moon is orange", but "The moon is orange" doesn't follow from any proper subset of A.

e. Which member of the following pairs of sentences expresses the more informative (that is, the stronger) proposition?

<**Mary has at least five lambs**, **Mary has at least 3 lambs**>

<**Mary has a lamb or a goat** (**or both**), **Mary has a lamb and a goat**>

<**Mary didn't sing or dance**, **Mary didn't sing and dance**>

<**Nobody has five lambs**, **nobody has three lambs**>

f. Can any two propositions be compared as to their informativeness?